Non-trivial intersecting uniform sub-families of hereditary families

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Abstract

For a family $\mathcal{F}$ of sets, let $\mu(\mathcal{F})$ denote the size of a smallest set in $\mathcal{F}$ that is not a subset of any other set in $\mathcal{F}$, and for any positive integer r, let $\mathcal{F}^{(r)}$ denote the family of r-element sets in $\mathcal{F}$. We say that a family $\mathcal{A}$ is of *Hilton-Milner* (*HM*) *type* if for some $A \in \mathcal{A}$, all sets in $\mathcal{A}\backslash\{A\}$ have a common element $x \notin A$ and intersect A. We show that if a *hereditary* family $\mathcal{H}$ is *compressed* and $\mu(\mathcal{H}) \geq 2r \geq 4$, then the HM-type family $\{A \in \mathcal{H}^{(r)} \colon 1 \in A, A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$ is a largest *non-trivial intersecting* sub-family of $\mathcal{H}^{(r)}$; this generalises a well-known result of Hilton and Milner. We demonstrate that for any $r \geq 3$ and $m \geq 2r$, there exist non-compressed hereditary families $\mathcal{H}$ with $\mu(\mathcal{H}) = m$ such that no largest non-trivial intersecting sub-family of $\mathcal{H}^{(r)}$ is of HM type, and we suggest two conjectures about the extremal structures for arbitrary hereditary families.

1 Introduction

In this introductory section we first set up the basic definitions and notation that are used throughout the paper, and then we give an overview of the known results that have inspired the work in this paper and present our main result.

1.1 Basic definitions and notation

Unless otherwise stated, we shall use small letters such as x to denote elements of a set or non-negative integers or functions, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (i.e. sets whose elements are sets themselves). It is to be assumed that arbitrary sets and families are *finite*.

We start with some notation for sets. $\mathbb{N}$ is the set $\{1, 2, ...\}$ of positive integers. For $m, n \in \mathbb{N}$ with $m \leq n$, the set $\{i \in \mathbb{N} \colon m \leq i \leq n\}$ is denoted by $[m, n]$, and if $m = 1$, then we also write $[n]$. For a set X, the *power set* $\{A\colon A \subseteq X\}$ of X is denoted by 2^X. We denote $\{Y \subseteq X\colon |Y| = r\}$, $\{Y \subseteq X\colon |Y| \leq r\}$ and $\{Y \subseteq X\colon |Y| < r\}$ by $\binom{X}{r}$, $\binom{X}{\leq r}$ and $\binom{X}{<r}$, respectively.

We next develop some notation for certain sets and families defined on a family $\mathcal{F} \subseteq 2^X$. Let $U(\mathcal{F})$ denote the union of all sets in $\mathcal{F}$. Let $\mathcal{F}^{(r)} := \{F \in \mathcal{F}\colon |F| = r\}$, $\mathcal{F}^{(\leq r)} := \{F \in \mathcal{F}\colon |F| \leq r\}$ and $\mathcal{F}^{(<r)} := \{F \in \mathcal{F}\colon |F| < r\}$. We call $\mathcal{F}^{(r)}$ the *r'th level of* $\mathcal{F}$. For $Y \subseteq X$, we set

$$\mathcal{F}(Y) := \{F \in \mathcal{F}\colon F \cap Y \neq \emptyset\},$$

and with slight abuse of notation, we set

$$\mathcal{F}(\overline{Y}) := \{F \in \mathcal{F}\colon F \cap Y = \emptyset\} = \mathcal{F} \backslash \mathcal{F}(Y).$$

For $x \in X$, we may abbreviate $\mathcal{F}(\{x\})$ and $\mathcal{F}(\overline{\{x\}})$ to $\mathcal{F}(x)$ and $\mathcal{F}(\overline{x})$, respectively, and we call $\mathcal{F}(x)$ a *star of* $\mathcal{F}$. We set

$$\mathcal{F}\langle x\rangle := \{F \backslash \{x\}\colon F \in \mathcal{F}(x)\} = \{E\colon x \notin E,\, E \cup \{x\} \in \mathcal{F}(x)\}.$$

To illustrate examples of families that can be defined on $\mathcal{F}$ with the above notation, we have

$$\mathcal{F}(x)(Y) = \{A \in \mathcal{F}(x)\colon A \cap Y \neq \emptyset\} = \{A \in \mathcal{F}\colon x \in A, A \cap Y \neq \emptyset\},$$
$$\mathcal{F}\langle x\rangle(Y) = \{A \in \mathcal{F}\langle x\rangle\colon A \cap Y \neq \emptyset\} = \{A\colon x \notin A,\, A \cup \{x\} \in \mathcal{F}(x),\, A \cap Y \neq \emptyset\}.$$

Let $i, j \in [n]$. We recall the *compression operation* $\Delta_{i,j}\colon 2^{2^{[n]}} \to 2^{2^{[n]}}$ (introduced in [12]) defined by

$$\Delta_{i,j}(\mathcal{F}) := \{\delta_{i,j}(F)\colon F \in \mathcal{F}, \delta_{i,j}(F) \notin \mathcal{F}\} \cup \{F \in \mathcal{F}\colon \delta_{i,j}(F) \in \mathcal{F}\},$$

where $\delta_{i,j}\colon 2^{[n]} \to 2^{[n]}$ is defined by

$$\delta_{i,j}(F) = \begin{cases} (F \backslash \{j\}) \cup \{i\} & \text{if } i \notin F \text{ and } j \in F; \\ F & \text{otherwise.} \end{cases}$$

The survey paper [13] outlines various uses of this operation in extremal set theory.

We now recall some basic definitions for families of sets. A set $F \in \mathcal{F}$ is said to be a *base of* $\mathcal{F}$ if F is not a subset of any other set in $\mathcal{F}$. We define

$$\mu(\mathcal{F}) := \min\{|F|\colon F \text{ is a base of } \mathcal{F}\}.$$

A family $\mathcal{F}$ is said to be
- a *hereditary family* (or an *ideal* or a *downset*) if all subsets of any set in $\mathcal{F}$ are in $\mathcal{F}$.
- *intersecting* if any two sets in $\mathcal{F}$ have at least one common element.
- *centred* if the sets in $\mathcal{F}$ have at least one common element.
- *non-centred*, or *non-trivial*, if $\mathcal{F}$ is not centred.
- *uniform* if all sets in $\mathcal{F}$ are of the same size (i.e. $\mathcal{F} = \mathcal{F}^{(r)}$ for some integer $r \geq 0$).

- an *antichain*, or a *Sperner family*, if all sets in $\mathcal{F}$ are bases of $\mathcal{F}$.
- *compressed with respect to* $x \in U(\mathcal{F})$ if $(F\backslash\{y\}) \cup \{x\} \in \mathcal{F}$ whenever $y \in F \in \mathcal{F}$ and $x \notin F$, i.e. if $\Delta_{x,y}(\mathcal{F}) = \mathcal{F}$ for any $y \in U(\mathcal{F})$.
- *compressed* if $\mathcal{F} \subseteq 2^{[n]}$ and $(F\backslash\{j\}) \cup \{i\} \in \mathcal{F}$ whenever $1 \leq i < j \in F \in \mathcal{F}$ and $i \notin F$, i.e. if $\mathcal{F} \subseteq 2^{[n]}$ and $\Delta_{i,j}(\mathcal{F}) = \mathcal{F}$ for any $i, j \in [n]$ with $i < j$.

A centred family is trivially intersecting. Also note that a star of a family is a maximal centred sub-family (where by maximal we mean that it is not a proper sub-family of another).

1.2 Old and new results

The following is a classical result in the literature on combinatorics of finite sets, known as the Erdős-Ko-Rado (EKR) Theorem.

Theorem 1.1 ([12]) *If $r \leq n/2$ and $\mathcal{A}$ is an intersecting sub-family of $\binom{[n]}{r}$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1},$$

and equality holds if $\mathcal{A}$ is a star of $\binom{[n]}{r}$.

Two alternative proofs that are particularly short and beautiful were obtained by Katona [21] and Daykin [10]. The seminal paper [12] (featuring the above theorem) inspired much research in extremal set theory, and this gave rise to many beautiful results; see [3, 11, 13].

In view of Theorem 1.1, we say that a family $\mathcal{F}$ has the *EKR property* if at least one of the largest intersecting sub-families of $\mathcal{F}$ is a star of $\mathcal{F}$, and that $\mathcal{F}$ has the *strict EKR property* if all the largest intersecting sub-families of $\mathcal{F}$ are stars of $\mathcal{F}$.

Hilton and Milner strengthened Theorem 1.1 as follows. For $2 \leq r \leq n/2$, let $\mathcal{N}_{n,r}$ denote the non-centred intersecting sub-family $\{A \in \binom{[n]}{r} : 1 \in A,\ A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$ of $\binom{[n]}{r}$.

Theorem 1.2 ([16]) *If $2 \leq r \leq n/2$ and $\mathcal{A}$ is a non-centred intersecting sub-family of $\binom{[n]}{r}$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1,$$

and equality holds if $\mathcal{A} = \mathcal{N}_{n,r}$.

Various alternative proofs of the above result were published; see, for example, [14, 15].

Note that Theorem 1.2 implies that, in addition to Theorem 1.1, if $r < n/2$, then an intersecting sub-family $\mathcal{A}$ of $\binom{[n]}{r}$ is of maximum size $\binom{n-1}{r-1}$ if and only if $\mathcal{A}$ is one of the stars $\{A \in \binom{[n]}{r} : i \in A\}$, $i = 1, \dots, n$, of $\binom{[n]}{r}$; that is, for $r < n/2$, $\binom{[n]}{r}$ has the strict EKR property. The problem for $n/2 < r \leq n$ is trivial. When $r = n$, $\binom{[n]}{r}$ consists only of the set $[n]$. If $n/2 < r < n$, then $\binom{[n]}{r}$ itself is a non-centred intersecting family. So $\binom{[n]}{r}$ has the EKR property if and only if $r \leq n/2$ or $r = n$.

We say that a family $\mathcal{A}$ is of *Hilton-Milner (HM) type* if for some $A \in \mathcal{A}$, all sets in $\mathcal{A}\backslash\{A\}$ have a common element $x \notin A$ and intersect A (so $\mathcal{A} = \mathcal{A}(x)(A) \cup \{A\}$). In view of Theorem 1.2, we say that a family $\mathcal{F}$ has the *HM property* if at least one of the largest non-centred intersecting sub-families of $\mathcal{F}$ is of HM type. Note that $\binom{[n]}{r}$ has the HM property if and only if $r \leq n/2$.

In this paper we are interested in the size or structure of largest non-centred intersecting sub-families of levels of hereditary families. Before coming to the main contribution in this paper, we shall outline more facts and provide further motivation.

Hereditary families are important combinatorial objects that have attracted much attention. The various interesting examples include the family of *independent sets* of a *graph* or *matroid*. The power set 2^X of a set X is the simplest example of a hereditary family. In fact, by definition, a family is hereditary if and only if it is a union of power sets. Note that if $X_1, ..., X_k$ are the bases of a hereditary family $\mathcal{H}$, then $\mathcal{H} = 2^{X_1} \cup ... \cup 2^{X_k}$.

One of the central problems in extremal set theory is Chvátal's conjecture [8], which claims that any hereditary family $\mathcal{H}$ is EKR. The best result so far on this conjecture is due to Snevily [23] and says that the conjecture is true if $\mathcal{H}$ is compressed with respect to an element ([4] provides a generalisation obtained by means of a self-contained alternative argument). Among others, this generalises a result of Chvátal [9] and another one of Schönheim [22]; the former verifies the conjecture for $\mathcal{H}$ compressed, and the latter verifies it for the case when the bases of $\mathcal{H}$ have a common element.

Generalising a conjecture of Holroyd and Talbot [19], the author [1] suggested the following uniform version of Chvátal's conjecture.

Conjecture 1.3 ([1]) *If $\mathcal{H}$ is a hereditary family and $r \leq \mu(\mathcal{H})/2$, then*
(i) $\mathcal{H}^{(r)}$ has the EKR property,
(ii) $\mathcal{H}^{(r)}$ has the strict EKR property if $r < \mu(\mathcal{H})/2$.

The Holroyd-Talbot conjecture claims that this holds when $\mathcal{H}$ is the family of all independent sets of a graph, and this has been verified for various classes of graphs [6, 7, 17, 18, 19, 20, 24]. A special case of the main result in [1] is that Conjecture 1.3 is true if $\mu(\mathcal{H}) \geq \frac{3}{2}(r-1)^2(3r-4)+r$. The same paper also features the following result.

Theorem 1.4 ([1]) *Conjecture 1.3 is true if $\mathcal{H}$ is compressed.*

Improving Theorem 1.4 to the case when $\mathcal{H}$ is compressed with respect to an element remains an open problem.

Conjecture 1.3 claims that the largest maximal centred sub-families (i.e. the stars) are optimal structures for $\mathcal{H}^{(r)}$ with $\mu(\mathcal{H}) \geq 2r$ and, as we mentioned above, this is indeed the case when $\mu(\mathcal{H})$ is sufficiently large (depending on r). As we specified above, the question we are dealing with in this paper is: What happens when we restrict the problem to intersecting sub-families of $\mathcal{H}^{(r)}$ that are not centred? Theorem 1.2 solves the problem for the case when $\mathcal{H}$ is $2^{[n]}$ and hence the level $\mathcal{H}^{(r)}$ is $\binom{[n]}{r}$. The condition $r \leq n/2$ is generalised by the condition $r \leq \mu(\mathcal{H})/2$; note that $\mu(\mathcal{H}) = n$ for $\mathcal{H} = 2^{[n]}$. Our main contribution is to generalise Theorem 1.2 to one for sub-families of compressed hereditary families using the compression method, exploiting the fact that if $\mathcal{F}$ is a compressed sub-family of $2^{[n]}$ and $\mathcal{A} \subseteq \mathcal{F}$, then $\Delta_{i,j}(\mathcal{A}) \subseteq \mathcal{F}$ for any $i, j \in [n]$ with $i < j$.

Theorem 1.5 *If $\mathcal{H}$ is a compressed hereditary sub-family of $2^{[n]}$ and $2 \leq r \leq \mu(\mathcal{H})/2$, then $\mathcal{H}^{(r)}$ has the HM property and, moreover, $\mathcal{H}^{(r)}(1)([2, r+1]) \cup \{[2, r+1]\}$ is a largest non-centred intersecting sub-family of $\mathcal{H}^{(r)}$.*

Part of the proof will be to show that $\mathcal{H}^{(r)}(1)([2, r+1]) \cup \{[2, r+1]\}$ is indeed a non-centred sub-family of $\mathcal{H}^{(r)}$. In the proof we take an approach that has some significant differences from any of those in the various proofs of Theorem 1.2 known to the author (particularly those we made reference to above).

An interesting fact revealed in the next section is that to have the HM property guaranteed, the condition in Theorem 1.5 of having $\mathcal{H}$ compressed not only cannot be removed, but also cannot be relaxed to having $\mathcal{H}$ compressed with respect to an element of $[n]$ or even having the bases share a common element. We then suggest two conjectures about the structure of the largest non-centred intersecting sub-families of $\mathcal{H}^{(r)}$ for any hereditary family $\mathcal{H}$ with $2 \leq r \leq \mu(\mathcal{H})/2$.

We conclude this section by showing that just like Theorem 1.2 yields Theorem 1.1, Theorem 1.5 yields Theorem 1.4.

Proof of Theorem 1.4 from Theorem 1.5. The result is trivial for $r = 1$, so we assume that $r \geq 2$. Since $\mathcal{H}$ is compressed, $[\mu(\mathcal{H})] \in \mathcal{H}$. Therefore,

$$\mathcal{H} \text{ compressed and hereditary} \Rightarrow 2^{[\mu(\mathcal{H})]} \subseteq \mathcal{H}. \tag{1}$$

Let $\mathcal{A}$ be a non-centred intersecting sub-family of $\mathcal{H}^{(r)}$, and let $\mathcal{N} = \mathcal{H}^{(r)}(1)([2, r+1]) \cup \{[2, r+1]\}$. By Theorem 1.5, $|\mathcal{A}| \leq |\mathcal{N}|$. Let $\mathcal{B} := \{B \in \binom{[\mu(\mathcal{H})] \backslash [2, r+1]}{r} \colon 1 \in B\}$. Clearly, $\mathcal{B} \cap \mathcal{N} = \emptyset$. By (1), $\mathcal{B} \subset \mathcal{H}^{(r)}(1)$. So

$$|\mathcal{H}^{(r)}(1)| \geq |(\mathcal{N} \backslash \{[2, r+1]\}) \cup \mathcal{B}| = |\mathcal{N}| - 1 + \binom{\mu(\mathcal{H}) - r - 1}{r - 1}.$$

Thus, since $r \leq \mu(\mathcal{H})/2$, we have $|\mathcal{A}| \leq |\mathcal{H}^{(r)}(1)|$ with strict inequality if $r < \mu(\mathcal{H})/2$. $\square$

2 A construction and two conjectures

We will now show that if the condition that $\mathcal{H}$ is compressed is removed from Theorem 1.5, then the assertion that $\mathcal{H}^{(r)}$ has the HM property is no longer true in general, except for the case $r = 2$ because a non-centred intersecting family of 2-element sets can only be of the form $\{\{a, b\}, \{a, c\}, \{b, c\}\}$. More precisely, for arbitrary $r \geq 3$ and $m \geq 2r$, we construct a hereditary family $\mathcal{H}$ such that $\mu(\mathcal{H}) = m$, the bases of $\mathcal{H}$ have common elements (and hence $\mathcal{H}$ is compressed with respect to each of these elements), and $\mathcal{H}^{(r)}$ does not have the HM property.

Proposition 2.1 *Let $k, l, p, r \in \mathbb{N}$ such that $3 \leq l \leq k$, $l \leq r \leq (k+l)/2$, $p\binom{k}{r-l+1} > \binom{k}{r-1} - \binom{k+l-r-1}{r-1} + 1$. Let $H_i := [l] \cup [(i-1)k+l+1, ik+l]$, $i = 1, ..., p$. Let $\mathcal{H} = \bigcup_{i=1}^{p} 2^{H_i}$. Then $\mathcal{H}$ is hereditary, $3 \leq r \leq (k+l)/2 = \mu(\mathcal{H})/2$, $[l]$ is a common subset of the bases of $\mathcal{H}$, and $\mathcal{H}^{(r)}$ does not have the HM property.*

Proof. Clearly, $\mathcal{H}$ is hereditary by definition, and $H_1, \ldots, H_p$ are the bases of $\mathcal{H}$. We have $H_{i_1} \cap H_{i_2} = [l]$ for any $i_1, i_2 \in [p]$. Let $m := \mu(\mathcal{H})$. So $m = |H_1| = \cdots = |H_p| = k + l \geq 2r$.

Let $\mathcal{A}_1$ be a sub-family of $\mathcal{H}^{(r)}$ of HM type. So $\mathcal{A}_1 := \mathcal{H}^{(r)}(a)(B) \cup \{B\}$ for some $B \in \mathcal{H}^{(r)}$ and $a \in [pk+l]\backslash B$. Let $\mathcal{N} := \mathcal{H}^{(r)}(1)([2, r+1]) \cup \{[2, r+1]\}$.

We first show that $|\mathcal{A}_1| \leq |\mathcal{N}|$. This is straightforward if $a \notin [l]$ because then $\mathcal{H}^{(r)}(a) \subset \binom{H_{i'}}{r}$ for some $i' \in [p]$. Suppose $a \in [l]$ instead, and let $j \in [p]$ such that $B \subset H_j$ (note that j is unique because, since $a \in [l]\backslash B$, we have $|B \cap [l]| \leq l - 1 \leq r - 1 < |B|$ and hence, for any $j' \in [p]\backslash\{j\}$, $B \nsubseteq H_{j'}$ since $H_j \cap H_{j'} = [l]$). Then

$$\begin{aligned}
|\mathcal{A}_1| &= \binom{|H_j| - 1}{r-1} - \binom{|H_j| - r - 1}{r-1} + 1 \\
&\quad + \sum_{i \in [p]\backslash\{j\}} \left(\binom{|H_i| - 1}{r-1} - \binom{|H_i| - |B \cap ([l]\backslash\{a\})| - 1}{r-1} \right) \\
&\leq p\binom{m-1}{r-1} - \binom{m-r-1}{r-1} + 1 - (p-1)\binom{m-l}{r-1} = |\mathcal{N}|.
\end{aligned}$$

Now let $\mathcal{A}_2 := \mathcal{H}^{(r)}(1)([l]\backslash\{1\}) \cup \{([l]\backslash\{1\}) \cup C \colon C \in \binom{H_i \backslash [l]}{r-l+1}$ for some $i \in [p]\}$. So $\mathcal{A}_2 \subset \mathcal{H}^{(r)}$. We have $|\mathcal{A}_2| = p\left(\binom{m-1}{r-1} - \binom{m-l}{r-1} + \binom{m-l}{r-l+1}\right)$ and hence

$$\begin{aligned}
|\mathcal{A}_2| - |\mathcal{N}| &= p\binom{m-l}{r-l+1} - \binom{m-l}{r-1} + \binom{m-r-1}{r-1} - 1 \\
&= p\binom{k}{r-l+1} - \binom{k}{r-1} + \binom{k+l-r-1}{r-1} - 1 > 0 \quad \text{(by choice of } p\text{)}.
\end{aligned}$$

So $|\mathcal{N}| < |\mathcal{A}_2|$. Therefore, since $\mathcal{A}_2$ is not of HM type and we established that $\mathcal{N}$ is a largest sub-family of $\mathcal{H}^{(r)}$ of HM type, the result follows. $\Box$

Having established that the largest sub-families of HM type are not always optimal, we now suggest two conjectures about the optimal structures in general.

Conjecture 2.2 (Weak Form) *If $\mathcal{H}$ is a hereditary family and $2 \leq r \leq \mu(\mathcal{H})/2$, then there exist $x, y \in U(\mathcal{H})$, $x \neq y$, such that $\mathcal{A} \subset \mathcal{H}^{(r)}(x) \cup \mathcal{H}^{(r)}(y)$ for some largest non-centred intersecting sub-family $\mathcal{A}$ of $\mathcal{H}^{(r)}$.*

Remark 2.3 Suppose Conjecture 2.2 is true, and let $\mathcal{A}$ be as in the conjecture. Since $\mathcal{A}$ is non-centred, at least one set in $\mathcal{A}$ contains x and does not contain y, and at least one set in $\mathcal{A}$ contains y and does not contain x. Since $\mathcal{A} \subset \mathcal{H}^{(r)}(x) \cup \mathcal{H}^{(r)}(y)$, all the sets in $\mathcal{H}^{(r)}$ containing both x and y intersect every set in $\mathcal{A}$, and hence all these sets must be in $\mathcal{A}$ because otherwise we can add to $\mathcal{A}$ all such sets that are not in $\mathcal{A}$ and still obtain a non-centred intersecting sub-family of $\mathcal{H}^{(r)}$. Summing up these observations, we have $\mathcal{A} \cap \mathcal{H}^{(r)}(x)(\overline{y}) \neq \emptyset$, $\mathcal{A} \cap \mathcal{H}^{(r)}(y)(\overline{x}) \neq \emptyset$ and $\mathcal{H}^{(r)}(x)(y) \subset \mathcal{A}$.

Conjecture 2.4 (Strong Form) *If $\mathcal{H}$ is a hereditary family and $2 \leq r \leq \mu(\mathcal{H})/2$, then there exist $Y \in \mathcal{H}$ with $1 \leq |Y| \leq r$ and $x \in U(\mathcal{H}) \backslash Y$ such that*

$$\{A \in \mathcal{H}^{(r)} \colon x \in A,\, A \cap Y \neq \emptyset\} \cup \{B \in \mathcal{H}^{(r)} \colon Y \subseteq B\}$$

is a largest non-centred intersecting sub-family of $\mathcal{H}^{(r)}$.

The family $\mathcal{A}_2$ in the proof of Proposition 2.1 has this structure.

Removing the condition $r \leq \mu(\mathcal{H})/2$ from the above conjectures gives false statements because, as we pointed out earlier, $\mu(2^{[n]}) = n$ and the whole level $\binom{[n]}{r}$ of $2^{[n]}$ is a non-centred intersecting family when $n/2 < r < n$. It is not clear what the extremal structures may be for $r > \mu(\mathcal{H})/2$.

3 Tools

In this section we state various known results that we will need to use in the proof of Theorem 1.5.

We start with the following well-known fundamental properties of compressions that emerged in [12] and that are not difficult to prove (see [13]).

Lemma 3.1 ([12]) *Let $\mathcal{A}$ be an intersecting sub-family of $2^{[n]}$.*
(i) For any $i, j \in [n]$, $\Delta_{i,j}(\mathcal{A})$ is intersecting.
(ii) If $r \leq n/2$, $\mathcal{A} \subseteq \binom{[n]}{r}$ and $\Delta_{i,n}(\mathcal{A}) = \mathcal{A}$ for all $i \in [n-1]$, then $(A \cap B) \backslash \{n\} \neq \emptyset$ for any $A, B \in \mathcal{A}$.

Our second important lemma purely concerns the parameter $\mu(\mathcal{F})$; the proofs of the parts of this lemma are scattered in [1] but collected in [5, Lemma 3.2].

Lemma 3.2 ([1, 5]) *Let $\emptyset \neq \mathcal{F} \subseteq 2^{[n]}$ and $a \in [n]$.*
(i) If $\mathcal{F}(a) \neq \emptyset$, then $\mu(\mathcal{F}\langle a \rangle) \geq \mu(\mathcal{F}) - 1$.
(ii) If $\mathcal{F}$ is hereditary, then $\mu(\mathcal{F}(\overline{a})) \geq \mu(\mathcal{F}) - 1$.
(iii) If $\mathcal{F}$ is compressed and $U(\mathcal{F}) = [n] \notin \mathcal{F}$, then $\mu(\mathcal{F}(\overline{n})) \geq \mu(\mathcal{F})$.

We shall say that a family $\mathcal{F} \subseteq 2^{[n]}$ is *quasi-compressed* if $\delta_{i,j}(F) \in \mathcal{F}$ for any $F \in \mathcal{F}$ and any $i, j \in U(\mathcal{F})$ with $i < j$. Therefore, a quasi-compressed family $\mathcal{F} \subseteq 2^{[n]}$ is isomorphic to a compressed sub-family of $2^{[|U(\mathcal{F})|]}$, and the isomorphism is induced by the bijection $\beta \colon U(\mathcal{F}) \to [|U(\mathcal{F})|]$ defined by $\beta(u_i) := i$, $i = 1, ..., |U(\mathcal{F})|$, where $\{u_1, ..., u_{|U(\mathcal{F})|}\} = U(\mathcal{F})$ and $u_1 < ... < u_{|U(\mathcal{F})|}$.

The next lemma is straightforward, so we omit its proof.

Lemma 3.3 *Let $\mathcal{H} \subseteq 2^{[n]}$ and let $a \in [n]$.*
(i) If $\mathcal{H}$ is hereditary, then $\mathcal{H}(\overline{a})$ and $\mathcal{H}\langle a\rangle$ are hereditary.
(ii) If $\mathcal{H}$ is quasi-compressed, then $\mathcal{H}(\overline{a})$ and $\mathcal{H}\langle a\rangle$ are quasi-compressed.

For a set $X := \{x_1, ..., x_n\} \subset \mathbb{N}$ with $x_1 < ... < x_n$ and $r \in [n]$, call $\{x_1, ..., x_r\}$ the *initial r-segment of X*. For convenience, we call $\emptyset$ the *initial 0-segment of X*.

Lemma 3.4 ([5]) *Let $\mathcal{F}$ be a quasi-compressed sub-family of $2^{[n]}$. Let $\emptyset \neq Z \subseteq [n]$ and let $Y \in \binom{[n]}{|Z|}$ such that Y contains the initial $|Z \cap U(\mathcal{F})|$-segment of $U(\mathcal{F})$. Then $|\mathcal{F}(Z)| \leq |\mathcal{F}(Y)|$.*

Two families $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-intersecting* if each set in $\mathcal{A}$ intersects each set in $\mathcal{B}$.

Theorem 3.5 ([5]) *If $r \leq s$, $n \geq r+s$, $\mathcal{H}$ is a compressed hereditary sub-family of $2^{[n]}$ with $\mu(\mathcal{H}) \geq r + s$, $\emptyset \neq \mathcal{A} \subset \mathcal{H}^{(r)}$, $\emptyset \neq \mathcal{B} \subset \mathcal{H}^{(s)}$, $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting, $\mathcal{A}_0 := \{[r]\}$ and $\mathcal{B}_0 := \mathcal{H}^{(s)}([r])$, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0| = 1 + |\mathcal{H}^{(s)}([r])|.$$

4 Proof of Theorem 1.5

We now work towards the proof of Theorem 1.5, using the tools in Section 3.

Lemma 4.1 *Let $\mathcal{H}$ be a compressed hereditary sub-family of $2^{[n]}$. Suppose $1 \leq p < q \leq n$ and $2 \leq r \leq \mu(\mathcal{H})/2$. Let $\mathcal{A}$ be a non-centred intersecting sub-family of $\mathcal{H}^{(r)}$ such that $\Delta_{p,q}(\mathcal{A})$ is centred. Then $|\mathcal{A}| \leq |\mathcal{H}^{(r)}(p)(I \cup \{q\})| + 1$, where I is the initial $(r-1)$-segment of $[n]\backslash\{p, q\}$.*

Proof. We are given that $\Delta_{p,q}(\mathcal{A}) \subseteq \mathcal{H}^{(r)}(a)$ for some $a \in [n]$. If $a \neq p$, then $\mathcal{A} \subseteq \mathcal{H}^{(r)}(a)$, contradicting $\mathcal{A}$ non-centred. So

$$\Delta_{p,q}(\mathcal{A}) \subseteq \mathcal{H}^{(r)}(p) \tag{2}$$

and hence $\mathcal{A} = \mathcal{A}(\{p, q\})$. So

$$|\mathcal{A}| = |\mathcal{A}(p)(q)| + |\mathcal{A}\langle p\rangle(\overline{q})| + |\mathcal{A}(\overline{p})\langle q\rangle| \tag{3}$$

and, since $\mathcal{A}$ is intersecting, $\mathcal{A}\langle p\rangle(\overline{q})$ and $\mathcal{A}(\overline{p})\langle q\rangle$ are cross-intersecting. $\mathcal{A}\langle p\rangle(\overline{q})$ and $\mathcal{A}(\overline{p})\langle q\rangle$ are also non-empty because otherwise $\mathcal{A} \subseteq \mathcal{H}^{(r)}(p)$ or $\mathcal{A} \subseteq \mathcal{H}^{(r)}(q)$ (contradicting $\mathcal{A}$ non-centred). Let $r' := r - 1$. Let $Z := [n]\backslash\{p, q\}$ and let $z_1 < ... < z_{n-2}$ such that $\{z_1, ..., z_{n-2}\} = Z$. Let $\mathcal{Z} := \mathcal{H}\langle p\rangle(\overline{q})$. So $\mathcal{Z} \subseteq 2^Z$. By (i) and (ii) of Lemma 3.2, $\mu(\mathcal{Z}) \geq \mu(\mathcal{H}) - 2$. Thus, since $r \leq \mu(\mathcal{H})/2$, we have $r' \leq (\mu(\mathcal{H}) - 2)/2 \leq \mu(\mathcal{Z})/2$. Since $\mathcal{H}$ is compressed and $p < q$, $\mathcal{H}(\overline{p})\langle q\rangle \subseteq \mathcal{Z}$. So $\mathcal{A}\langle p\rangle(\overline{q}), \mathcal{A}(\overline{p})\langle q\rangle \subset \mathcal{Z}^{(r')}$. By Lemma 3.3, $\mathcal{Z}$ is hereditary and quasi-compressed. We note that, moreover, $\Delta_{i,j}(\mathcal{Z}) = \mathcal{Z}$ for any $i, j \in Z$

with $i < j$ (that is, $\mathcal{Z}$ is isomorphic to a compressed sub-family of $2^{[n-2]}$). Thus, from Theorem 3.5 we obtain

$$|\mathcal{A}\langle p\rangle(\overline{q})| + |\mathcal{A}(\overline{p})\langle q\rangle| \leq |\mathcal{Z}^{(r')}(I)| + 1.$$

Together with (3) this gives us

$$|\mathcal{A}| \leq |\mathcal{H}^{(r)}(p)(q)| + |\mathcal{Z}^{(r')}(I)| + 1 = |\mathcal{H}^{(r)}(p)(q)| + |\mathcal{H}^{(r)}(p)(\overline{q})(I)| + 1 = |\mathcal{H}^{(r)}(p)(I \cup \{q\})| + 1$$

as required. □

Lemma 4.2 *Let $\mathcal{F}$ be a compressed sub-family of $2^{[n]}$. Suppose $B \in \mathcal{F}^{(r)}$, $2 \leq r < n$, and $a \in [n] \backslash B$. Then $|\mathcal{F}^{(r)}(a)(B)| \leq |\mathcal{F}^{(r)}(1)([2, r+1])|$.*

Proof. Let $\mathcal{B} := \mathcal{F}^{(r)}(a)(B)$. Since $\mathcal{F}$ is compressed, we clearly have $\Delta_{1,a}(\mathcal{B}) \subseteq \mathcal{F}^{(r)}(1)(C)$, where

$$C = \begin{cases} (B \backslash \{1\}) \cup \{a\} & \text{if } a \neq 1 \in B; \\ B & \text{if } a = 1 \text{ or } 1 \notin B. \end{cases}$$

It is easy to see that having $\mathcal{F}$ compressed, $\mathcal{F}^{(r)} \neq \emptyset$ and $2 \leq r < n$ implies that $\mathcal{F}^{(r)}\langle 1\rangle$ is quasi-compressed and $U(\mathcal{F}^{(r)}\langle 1\rangle) = [2, m]$, $m = \min\{k \in [2, n] \colon \mathcal{F}^{(r)} \subseteq 2^{[k]}\}$. So $|\mathcal{F}^{(r)}\langle 1\rangle(C)| \leq |\mathcal{F}^{(r)}\langle 1\rangle([2, r+1])|$ by Lemma 3.4. Since $1 \notin C$, $|\mathcal{F}^{(r)}\langle 1\rangle(C)| = |\mathcal{F}^{(r)}(1)(C)|$. So we have $|\mathcal{B}| = |\Delta_{1,a}(\mathcal{B})| \leq |\mathcal{F}^{(r)}(1)(C)| \leq |\mathcal{F}^{(r)}(1)([2, r+1])|$. □

Our last lemma is based on the idea of the general problem in [2] and generalises one of the various results in that paper.

Lemma 4.3 *Let $\mathcal{H}$ be a compressed hereditary sub-family of $2^{[n]}$ with $\mathcal{H}(n) \neq \emptyset$. Suppose $2 \leq r \leq \mu(\mathcal{H})/2$. Let $\mathcal{A}$ be a compressed intersecting sub-family of $\mathcal{H}^{(r)}$. Let $r+1 \leq k \leq n$, $K := [2, k]$, $\mathcal{K}_1 := \mathcal{H}^{(r)}(1)([2, r+1]) \cup \{[2, r+1]\}$ and $\mathcal{K}_2 := \mathcal{H}^{(r)}(1)$.*
(i) If $k = r+1$, then $|\mathcal{A}(K)| \leq |\mathcal{K}_1(K)|$ $(= |\mathcal{K}_1|)$.
(ii) If $k \geq r+2$, then $|\mathcal{A}(K)| \leq |\mathcal{K}_2(K)|$.

Proof. Suppose $r = n/2$. Then $\mu(\mathcal{H}) = 2r$ and hence $[2r] \in \mathcal{H}$. So $\mathcal{H}^{(r)} = \binom{[2r]}{r}$ (as $\mathcal{H}$ is hereditary). For every $A \in \binom{[2r]}{r}$, the complement $[2r] \backslash A$ of A is the unique set in $\binom{[2r]}{r}$ that does not intersect A. So $|\mathcal{A}| \leq \frac{1}{2}\binom{2r}{r} = |\mathcal{K}_1(K)|$. Since in this case we have $|\mathcal{K}_1(K)| = |\mathcal{K}_1| = |\mathcal{K}_2| = |\mathcal{K}_2(K)|$ for $k \geq r+2$, (i) and (ii) follow.

Given that $\mathcal{A}$ is a compressed intersecting family, it is trivial that if $r = 2$ then $\mathcal{A} = \binom{[3]}{2}$ or $\mathcal{A} \subseteq \mathcal{H}^{(2)}(1)$, depending on whether $\mathcal{A}$ is non-centred or centred respectively. So the result for $r = 2$ is easy to check.

We now consider $3 \leq r < n/2$ and proceed by induction on n. Let $n' := n-1$ and $r' := r-1$. By Lemma 3.3, $\mathcal{H}(\overline{n})$ and $\mathcal{H}\langle n\rangle$ are compressed hereditary sub-families of $2^{[n']}$. We have $\mathcal{A}(\overline{n}) \subset \mathcal{H}(\overline{n})^{(r)}$ and $\mathcal{A}\langle n\rangle \subset \mathcal{H}\langle n\rangle^{(r')}$. $\mathcal{A}(\overline{n})$ is intersecting as $\mathcal{A}(\overline{n}) \subseteq \mathcal{A}$. Having $\mathcal{A}$ compressed means that the conditions of Lemma 3.1(ii) are satisfied and hence $\mathcal{A}\langle n\rangle$ is intersecting. Since $\mathcal{H}(n) \neq \emptyset$ and $r \leq \mu(\mathcal{H})/2$, it follows by Lemma 3.2(i) that

$r' < \mu(\mathcal{H}\langle n\rangle)/2$. If $[n] \in \mathcal{H}$, then $n = \mu(\mathcal{H}) = \mu(\mathcal{H}(\overline{n})) + 1$, and hence $r \leq \mu(\mathcal{H}(\overline{n}))/2$ as $r < n/2$. If $[n] \notin \mathcal{H}$ instead, then $r \leq \mu(\mathcal{H}(\overline{n}))/2$ follows from Lemma 3.2(iii) and $r \leq \mu(\mathcal{H})/2$. Thus, by the inductive hypothesis,

$$|\mathcal{A}(\overline{n})(K)| \leq |\mathcal{K}(\overline{n})(K)| \text{ and } |\mathcal{A}\langle n\rangle(K)| \leq |\mathcal{K}_2\langle n\rangle(K)|,$$

where

$$\mathcal{K} = \begin{cases} \mathcal{K}_1 & \text{if } k = r+1; \\ \mathcal{K}_2 & \text{if } k \geq r+2. \end{cases}$$

It is clear that we therefore have

$$|\mathcal{A}(K)| = |\mathcal{A}(\overline{n})(K)| + |\mathcal{A}\langle n\rangle(K)| \leq |\mathcal{K}(\overline{n})(K)| + |\mathcal{K}_2\langle n\rangle(K)| = |\mathcal{K}(K)|,$$

and hence (i) and (ii). □

Proof of Theorem 1.5. We may assume that $\mathcal{H}(n) \neq \emptyset$ because otherwise we can replace n by $m := \max\{k \in [n] : \mathcal{H}(k) \neq \emptyset\}$ since $\mathcal{H} \subseteq 2^{[m]}$. Let $\mathcal{N} := \mathcal{H}^{(r)}(1)([2, r+1]) \cup \{[2, r+1]\}$.

By (1) and the given condition that $\mu(\mathcal{H}) \geq 2r$, we have $[2, r+1] \in 2^{[2r]} \subset \mathcal{H}$. So $\mathcal{N}$ is a non-centred intersecting sub-family of $\mathcal{H}^{(r)}$.

Let $\mathcal{A}$ be a non-centred intersecting sub-family of $\mathcal{H}^{(r)}$. We apply compressions $\Delta_{i,j}$ with $i < j$ to $\mathcal{A}$ until a compressed family $\mathcal{A}^*$ is obtained (it is well-known and easy to see that such a procedure indeed takes a finite number of steps). $\mathcal{A}^*$ is intersecting by Lemma 3.1(i), and $\mathcal{A}^* \subset \mathcal{H}^{(r)}$ as $\mathcal{H}$ is compressed. Clearly, a compression does not alter the size of a family, so $|\mathcal{A}^*| = |\mathcal{A}|$.

Suppose $\mathcal{A}^*$ is centred. By Lemma 4.1, $|\mathcal{A}^*| \leq |\mathcal{H}^{(r)}(p^*)(I \cup \{q^*\})| + 1$ for some $p^*, q^* \in [n]$, $p^* < q^*$, where I is the initial $(r-1)$-segment of $[n] \backslash \{p^*, q^*\}$. Thus, by Lemma 4.2, $|\mathcal{A}^*| \leq |\mathcal{H}^{(r)}(1)([2, r+1])| + 1 = |\mathcal{N}|$.

Now suppose $\mathcal{A}^*$ is non-centred. Then $[2, r+1] \in \mathcal{A}^*$ as $\mathcal{A}^*$ is compressed. Thus, since $\mathcal{A}^*$ is intersecting, we have $\mathcal{A}^* = \mathcal{A}^*([2, r+1])$ and, by Lemma 4.3(i),

$$|\mathcal{A}^*([2, r+1])| \leq |\mathcal{N}([2, r+1])|. \tag{4}$$

Since $|\mathcal{A}| = |\mathcal{A}^*|$ and $\mathcal{N}([2, r+1]) = \mathcal{N}$, the result follows. □

5 An extension of Theorem 1.5 for Sperner sub-families

For any pair of families $\mathcal{A}$ and $\mathcal{F}$, let

$$\partial_{\mathcal{F}}^{(s)} \mathcal{A} := \{F \in \mathcal{F}^{(s)} : \text{ there exists } A \in \mathcal{A} \text{ such that either } A \subseteq F \text{ or } F \subseteq A\}.$$

The following inequality, which is the cornerstone of the main result in [1], enables us to extend Theorem 1.5 to one for Sperner sub-families.

Lemma 5.1 ([1]) *If $\mathcal{H}$ is a hereditary family, $r \leq s \leq \mu(\mathcal{H})$ and $\mathcal{A} \subseteq \mathcal{H}^{(r)}$, then*

$$|\partial_{\mathcal{H}}^{(s)} \mathcal{A}| \geq \frac{\binom{\mu(\mathcal{H})-r}{s-r}}{\binom{s}{s-r}} |\mathcal{A}|.$$

We mention in passing that an immediate important consequence of this inequality is that by taking $\mathcal{A} = \mathcal{H}^{(r)}$ we obtain $|\mathcal{H}^{(s)}| \geq \frac{\binom{\mu(\mathcal{H})-r}{s-r}}{\binom{s}{s-r}}|\mathcal{H}^{(r)}|$ ([1, Corollary 3.2]). However, Lemma 5.1 also has the following consequence ([1, Corollary 3.4]).

Corollary 5.2 ([1]) *If $\mathcal{H}$ is a hereditary family, $r \leq \mu(\mathcal{H})/2$, and $\mathcal{A}$ is a Sperner sub-family of $\mathcal{H}^{(\leq r)}$ such that $\mathcal{A} \cap \mathcal{H}^{(<r)} \neq \emptyset$, then $|\partial_{\mathcal{H}}^{(r)}\mathcal{A}| > |\mathcal{A}|$.*

It follows that if $\mathcal{A}$ is a largest intersecting Sperner sub-family of $\mathcal{H}^{(\leq r)}$, then $\mathcal{A} \subset \mathcal{H}^{(r)}$ ([1, Corollary 3.5]). We now show that Corollary 5.2 gives us the same result when we restrict ourselves to intersecting Sperner sub-families that are non-centred.

Corollary 5.3 *If $\mathcal{H}$ is a hereditary family, $r \leq \mu(\mathcal{H})/2$, and $\mathcal{A}$ is a largest non-centred intersecting Sperner sub-family of $\mathcal{H}^{(\leq r)}$, then $\mathcal{A} \subset \mathcal{H}^{(r)}$.*

Proof. Suppose $\mathcal{A} \cap \mathcal{H}^{(<r)} \neq \emptyset$. Let $\mathcal{A}^* := \partial_{\mathcal{H}}^{(r)}\mathcal{A}$. Since $\mathcal{A}^*$ is an intersecting Sperner sub-family of $\mathcal{H}^{(r)}$ and $|\mathcal{A}^*| > |\mathcal{A}|$ by Corollary 5.2, $\mathcal{A}^*$ must be centred. Let $a \in \bigcap_{A \in \mathcal{A}^*} A$. Since $\mathcal{A}$ is non-centred, $a \notin A'$ for some $A' \in \mathcal{A}$. Suppose $|A'| = r$. Then $A' \in \mathcal{A}^*$, but this contradicts $\mathcal{A}^* = \mathcal{A}^*(a)$. So $|A'| < r$. Let M be some maximal set in $\mathcal{H}$ such that $A' \subset M$. Since $|A'| < r \leq \mu(\mathcal{H})/2 \leq |M|/2 \leq |M \backslash \{a\}|$ and $\mathcal{H}$ is hereditary, there exists $A'' \in \mathcal{H}$ such that $A' \subset A'' \subseteq M \backslash \{a\}$ and $|A''| = r$. So $a \notin A'' \in \mathcal{A}^*$, contradicting $\mathcal{A}^* = \mathcal{A}^*(a)$. Therefore, $\mathcal{A} \cap \mathcal{H}^{(<r)} = \emptyset$ and hence the result. □

Theorem 1.1 was actually proved for Sperner sub-families of $\binom{[n]}{\leq r}$; more precisely, it was shown in [12] that an intersecting Sperner sub-family $\mathcal{A}$ of $\binom{[n]}{\leq r}$ can be of maximum size only if $\mathcal{A} \subseteq \binom{[n]}{r}$, and hence, by Theorem 1.1, the maximum size is $\binom{n-1}{r-1}$. Theorem 1.2 was actually proved in [16] in this more general form too, and this is immediately generalised by Theorem 1.5 and Corollary 5.3 as follows.

Theorem 5.4 *If $\mathcal{H}$ is a compressed hereditary sub-family of $2^{[n]}$ and $2 \leq r \leq \mu(\mathcal{H})/2$, then $\mathcal{H}^{(r)}(1)([2, r+1]) \cup \{[2, r+1]\}$ is a largest non-centred intersecting Sperner sub-family of $\mathcal{H}^{(\leq r)}$.*

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